An Analytic Framework for JavaScript

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Abstract

As the programming language of the web, JavaScript deserves a principled yet robust framework for static analysis. To achieve both aims simultaneously, we start from an established reduction semantics for JavaScript and systematically derive its intensional abstract interpretation.

Our first step is to transform the semantics into an equivalent low-level abstract machine: the JavaScript Abstract Machine (JAM). We then derive the systematic abstraction of the entire low-level machine. That process yields a finite-state, machine-based abstract interpretation for JavaScript. The calculation of this analysis is itself a milestone, constituting the first "field validation" of the theory behind systematically abstracting abstract machines. This finite-state framework allows us to import important techniques from the over 30 years of work on higher-order program analysis. We can instantiate the abstraction to obtain traditional analyses, such as k-CFA and CPA, extended to JavaScript.

Not content with the precision of this analysis over complex control effects, we extend our systematic approach with a new mode: unbounded abstraction of continuations. This new mode yields an infinite-state yet decidable pushdown machine whose stack precisely models the structure of the concrete program stack. The precise model of stack structure in turn confers *precise* control-flow analysis over control effects, such as exceptions, finally blocks, and of course, calls and returns. Both the finite-state and pushdown frameworks for abstract interpretation are sound and computable.

Categories and Subject Descriptors F.3.2 [*Logics and Meanings of Programs*]: Semantics of Programming Languages—Program analysis, Operational semantics

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1. Introduction

JavaScript is the dominant language of the web, making it the most available programming language in use today. Beyond the browser, it is increasingly important as a general-purpose language, as a server-side scripting language, and as an embedded scripting language.[1] Due to its ubiquity, JavaScript has become the target language for an array of compilers for languages such as C#, Java, Ruby, and others, making JavaScript a widely used "assembly language." As JavaScript cements its foundational role, the importance of robust static reasoning tools for that foundation grows.

However, JavaScript is an expressive, aggressively dynamic, high-level programming language. It is a higher-order, imperative, untyped language that is both functional and object-oriented, with prototype-based inheritance, constructors, non-local control, and a number of semantic quirks. Most quirks simply demand attention to detail, e.g.:

```
if (false) { var x ; }
... x ... // x is defined
```

Other quirks, such as the much-maligned `with` construct end up succumbing to an unremarkable desugaring. Yet other features, like non-local control effects and prototypical inheritance, require attention in the mechanics of the analysis itself; for a hint of what is possible, consider:

```
out: while (true)
 try {
   break out ;
 } finally {
   try {
     return 10 ;
   } finally {
     console.log("this runs; 10 returns") ;
   }
 }
```

It has become customary when reasoning about JavaScript to assume well-behavedness—that some subset of its features are never (or rarely) used for many programs. Richards, Lebresne, Burg and Vitek's thorough study [34] has cast em-

[1] Notably, Java 6 includes support for scripting applications via the `javax.script` package, and the JDK ships with the Mozilla Rhino JavaScript engine.

pirical doubt on these well-behavedness assumptions, finding almost every language feature used in almost every program in a large corpus of widely deployed JavaScript code.

Our goal is a principled approach for reasoning about *all* of JavaScript, including its unusual semantic peculiarities and its complex control mechanisms. To make this possible, the first step is the calculation of an *abstractable* abstract machine from an established semantics for JavaScript. From there, a finite-state abstract interpretation of that abstract machine yields a sound, robust framework for static analysis of JavaScript.

Motivated by the desire to handle non-local control effects such as exceptions and `finally` precisely, we will depart from standard practice in higher-order program analysis to derive an *infinite*-state yet decidable pushdown abstraction from our original abstract machine. The stack of the pushdown abstract interpretation *exactly* models the stack of the original abstract machine with no loss of structure—approximation is inflicted on only the control states. This pushdown framework offers a degree of precision in reasoning about control inaccessible to previous analyzers.

Contributions

The primary contributions of this paper include:

1. a provably correct abstract machine for all of JavaScript, sans `eval`,
2. a provably sound and computable framework for finite-state abstract interpretations of JavaScript programs, and
3. a provably sound and computable framework for infinite-state pushdown abstract interpretations that *exactly* models the program stack.

The first enables new avenues of research on intensional aspects of JavaScript execution such as space-consumption properties like proper tail recursion [8], stack-inspection [41], their combination [7], and compile-time approximation [39].

The second enables the import and application of traditional approaches to program analysis such as k-CFA [36] and the Cartesian-product algorithm (CPA) [2], as well as more modern analysis innovations such as abstract garbage-collection and strong-update [28, 19], all of which can be incorporated in a straightforward manner due to our machine-based approach.

The third enables *precise* reasoning about complex local and non-local control-flow relationships, such as proper matching between calls and returns, throws and handlers, and breaks and labels.[2] To model non-local control precisely, the analysis must model the program stack precisely. Yet, the program stack can grow without bound—a substantial obstacle for the finite-state framework. Pushdown abstraction maps that program stack onto the unbounded stack of a pushdown system. Because the analysis inflicts a finite-state abstraction on the *control states* of the pushdown system, the analysis remains decidable.

In support of our primary contributions, our secondary contributions include the development of a variant of a known formal model for JavaScript as a calculus of explicit substitutions; a detailed derivation, carried out in SML, going from our calculus to the JavaScript abstract machine via small, meaning-preserving program transformations; and an executable semantic models for the reduction semantics, the abstract machine and both its finite- and pushdown-abstractions, written in PLT Redex [15].

Background and notation: We assume a basic familiarity with reduction semantics and abstract machines. For background on concepts, terminology, and notation employed in this paper, we refer the reader to *Semantics Engineering with PLT Redex* [15]. Our construction of machines from reduction semantics follows Danvy, *et al.*'s refocusing-based approach [13, 4, 12]. Finally, for background on systematic abstract interpretation of classical abstract machines, see our recent work on the approach [39].

2. A calculus of explicit substitutions: $\lambda\rho_{JS}$

Our semantics-based approach to analysis is founded on abstract machines, which give an idealized characterization of a low-level language implementation. As such, we need a correct abstract machine for JavaScript. Rather than design one from scratch and manually verify its correctness after the fact, we rely on the syntactic correspondence between calculi and machines and adopt the refocusing-based approach of Danvy, *et al.*, to construct an abstract machine systematically from an established semantics for JavaScript.

To derive a machine, we must first choose a semantics from which to begin. The JavaScript standard [1] is large and informal and, therefore, unsuitable. There have been two major efforts in formalizing semantics adequate for all or most of JavaScript.

Maffeis, Mitchell, and Taly give a small-step structured operational semantics for all of JavaScript [25] as defined in the ECMA-262 Standard, 3rd Edition. The semantics aims to directly model features in a way that corresponds closely to common understandings and the informal specifications of how the language operates. So for example, JavaScript scope objects [1, Section 10.1.4], which are JavaScript objects representing activation records, are heap-allocated structures. The full semantics is about 70 pages.

Guha, Saftoiu, and Krishnamurthi [17] give a much smaller core calculus, λ_{JS}, with a small-step reduction semantics using evaluation contexts and demonstrate that full JavaScript can be desugared into λ_{JS}. The semantics accounts for all of JavaScript's features with the exception of

[2] One might wonder why `break` to a label requires non-local reasoning. In fact, it should not require it, but Guha *et al.*'s desugaring into λ_{JS}, handles `break` using powerful escape continuations. Constraints in the desugaring process prevent these escape continuations from crossing interprocedural boundaries, but unrestricted—or optimized λ_{JS}—may violate these constraints. For completeness, we handle full, unrestricted λ_{JS}, which means we must model these general escape continuations.

$$
\begin{array}{rcl}
s & \in & \mathit{String} \\
n & \in & \mathit{Number} \\
a & \in & \mathit{Address} \\
x & \in & \mathit{Variable} \\
e & ::= & x \mid s \mid n \mid a \mid \mathsf{true} \mid \mathsf{false} \mid \mathsf{undef} \mid \mathsf{null} \\
& \mid & \mathsf{fun}(\overline{x})\ \{\ e\ \} \mid \{\ \overline{s : e}\ \} \\
& \mid & \mathsf{let}\ (x = e)\ e \mid e(\overline{e}) \mid e[e] \\
& \mid & e[e] = e \mid \mathsf{del}\ e[e] \mid e = e \mid \mathsf{ref}\ e \mid \mathsf{deref}\ e \\
& \mid & \mathsf{if}\ (e)\ \{\ e\ \}\{\ e\ \} \mid e;\ e \mid \mathsf{while}(e)\ \{\ e\ \} \\
& \mid & \ell : \{\ e\ \} \mid \mathsf{break}\ \ell\ e \mid \mathsf{try}\ \{\ e\ \}\ \mathsf{catch}\ (x)\ \{\ e\ \} \\
& \mid & \mathsf{try}\ \{\ e\ \}\ \mathsf{finally}\ \{\ e\ \} \mid \mathsf{throw}\ e \mid \mathit{op}(\overline{e})
\end{array}
$$

Figure 1: Syntax of λ_{JS}

`eval`. Only some of JavaScript quirks are modeled directly, while other aspects are treated traditionally. For example, lexical scope is modeled with substitution. The desugarer is modeled formally and also available as a standalone Haskell program.

We choose to adopt the λ_{JS} model since its small size results in a tractably sized abstract machine. In principle, the larger, more direct semantics of Maffeis, *et al.*, could be used as a basis for a machine construction and subsequent abstract interpretation, and the directness of the semantics may yield fruitful analytic information, but the non-traditional aspects of the semantics pose challenges for the syntactic correspondence beyond the scope of this research.

The remainder of this paper focuses on machines and abstract interpretation for λ_{JS}. We refer the reader to Guha, *et al.*, for details on desugaring JavaScript to λ_{JS} and rational for the design decisions made.

2.1 Syntax

The syntax of λ_{JS} is given in figure 1. Following the convention of Guha, *et al.*, we use a sans-serif font to write λ_{JS} syntax and a fixed-width typewriter font for JavaScript.

Syntactic constants include strings, numbers, addresses, booleans, the undefined value, and the null value. Addresses are first-class values used to model mutable references. Heap allocation and dereference is made explicit through desugaring to λ_{JS}. Syntactic values include constants, function terms, and records. Records are keyed by strings and operations on records are modeled by functional update, extension, and deletion. Expressions include variables, syntactic values, and syntax for let binding, function application, record dereference, record update, record deletion, assignment, allocation, dereference, conditionals, sequencing, while loops, labels, breaks, exception handlers, finalizers, exception raising, and application of primitive operations. A program is a closed expression.

$$
\begin{array}{rcl}
v & ::= & s \mid n \mid a \mid \mathsf{true} \mid \mathsf{false} \mid \mathsf{undef} \mid \mathsf{null} \\
& \mid & (\mathsf{fun}(\overline{x})\ \{\ e\ \}, \rho) \mid \{\ \overline{s : v}\ \} \\
c & ::= & (e, \rho) \mid \{\ \overline{s : c}\ \} \mid c(\overline{c}) \mid \mathsf{let}\ (x = c)\ c \mid c[c] \\
& \mid & c[c] = c \mid \mathsf{del}\ c[c] \mid c = c \mid \mathsf{ref}\ c \mid \mathsf{deref}\ c \\
& \mid & \mathsf{if}\ (c)\ \{\ c\ \}\{\ c\ \} \mid c;\ c \mid \mathsf{while}(c)\ \{\ c\ \} \\
& \mid & \ell : \{\ c\ \} \mid \mathsf{break}\ \ell\ c \mid \mathsf{try}\ \{\ c\ \}\ \mathsf{catch}\ (x)\ \{\ c\ \} \\
& \mid & \mathsf{try}\ \{\ c\ \}\ \mathsf{finally}\ \{\ c\ \} \mid \mathsf{throw}\ c \mid \mathit{op}(\overline{c})
\end{array}
$$

Figure 2: Syntax of $\lambda\rho_{JS}$

2.2 Semantics

Guha, *et al.*, give a substitution-based reduction semantics formulated in terms of Felleisen-Hieb-style evaluation contexts [15]. The use of substitution in λ_{JS} is traditional from a theoretical point of view, and is motivated in part by want of conventional reasoning techniques such as subject reduction. On the other hand, environments are traditional from an implementation point of view. To mediate the gap, we first develop a variant of λ_{JS} that models the meta-theoretic notion of substitution with explicit substitutions.

Substitutions are represented explicitly with *environments*, which are finite maps from variables to values. We view environments as both finite sets and functions:

$$\rho = \{(x_1, v_1), \ldots, (x_n, v_n)\}, \text{ where for all } i \neq j,\ x_i \neq x_j.$$

Functional update, $\rho[x \mapsto v]$, is defined as

$$\rho[x \mapsto v](y) = \begin{cases} v & \text{if } x = y \\ \rho(y) & \text{if } x \neq y \end{cases}$$

Substitution $[v/x]e$, which is a meta-theoretic notation denoting e with all free-occurrences of x replaced by v, is represented at the syntactic level in $\lambda\rho_{JS}$, a calculus of explicit substitutions, as a pair consisting of e and an environment representing the substitution: $(e, \{(x, v)\})$. Such a pair is known as a *closure*.

The heap is modeled as a top-level *store*, a finite map from addresses to values:

$$\sigma = \{(a_1, v_1), \ldots, (a_n, v_n)\}, \text{ where for all } i \neq j,\ a_i \neq a_j.$$

The complete syntax of values and closures in $\lambda\rho_{JS}$ is given in figure 2. The semantics of $\lambda\rho_{JS}$ is given in terms of a small-step reduction relation defined in figure 4. There are three classes of reductions:

1. reductions that propagate environments from closures to inner expressions,
2. context-insensitive, store-insensitive reductions that operate over closures to implement computations that have no effect on the context or store, and
3. context-sensitive or store sensitive reductions that operate over pairs of stores and programs.

There is a fourth class of reductions, which we have omitted for space, which deal with raising exceptions in the case of run-time errors such as applying a non-function, branching on a non-boolean, indexing into a non-record or with a non-string key, etc. In each case, the faulty closure reduces to raising an exception with an error message. For example, the omitted reductions include:[3]

$$\textsf{if } (v) \{\ c_0\ \}\{\ c_1\ \} \rightarrow \textsf{throw } \text{“if: not a boolean”}, \quad \text{if } v \text{ is not } \textsf{true} \text{ or } \textsf{false}.$$

As a result, $\lambda\rho_{JS}$ programs do not get stuck: either they diverge or result in a value or an uncaught exception.

Reduction proceeds by a program being decomposed into a redex and evaluation context, which represents a portion of program text with a single hole, written [ ]. The grammar of evaluation contexts, defined in figure 3, specifies *where* in a program reduction may occur. The notation $\mathcal{E}[c]$ denotes both the decomposition of a program into the evaluation context $\mathcal{E}$ with c in the hole and the plugging of c into $\mathcal{E}$, which replaces the single hole in $\mathcal{E}$ by c. In addition to closures, holes may also be replaced by contexts, which yields another context. This is indicated with the notation $\mathcal{E}[\mathcal{E}']$.

There are three classes of evaluation contexts in figure 3: *local contexts* $\mathcal{C}$ range over all contexts that do not include exception handlers, finalizers, or labels; *control contexts* $\mathcal{D}$ range over contexts that are either empty or have a outermost exception handler, finalizer, or label; and *general contexts* $\mathcal{E}$ range over all evaluation contexts.

The distinction is made to describe the behavior of $\lambda\rho_{JS}$'s control constructs: breaks, finalizers, and exceptions. When an exception is thrown, the enclosing local context is discarded and if the nearest enclosing control context is an exception handler, the thrown value is given to the handler:[4]

$$\mathcal{E}[\textsf{try } \{\ \mathcal{C}[\textsf{throw } v]\ \} \textsf{ catch } (x)\ \{\ (e, \rho)\ \}] \longmapsto \mathcal{E}[(e, \rho[x \mapsto v])].$$

If the nearest enclosing control context is a finalizer, the finalization clause is evaluated and the exception is rethrown:

$$\mathcal{E}[\textsf{try } \{\ \mathcal{C}[\textsf{throw } v]\ \} \textsf{ finally } \{\ c\ \}] \longmapsto \mathcal{E}[c; \textsf{ throw } v].$$

If the nearest enclosing control context is a label, the context up to and including the label are discarded and the thrown value continues outward toward the next control context:

$$\mathcal{E}[\ell : \{\ \mathcal{C}[\textsf{throw } v]\ \}] \longmapsto \mathcal{E}[\textsf{throw } v].$$

Finally, if there is no enclosing control context, the exception was not handled and the program has resulted in an error:

$$\mathcal{C}[\textsf{throw } v] \longmapsto \textsf{err } v.$$

[3] One of JavaScript's quirks are its broad definitions of which values act as true and false, a quirk which doesn't appear to be modeled here at first glance. The desugaring transformation eliminates this quirk by coercing the condition in an **if** expression.

[4] We omit the store from these examples since they have no effect upon it.

$$\begin{array}{rcll}
\mathcal{C} & ::= & [\,] & (1)\\
& | & \textsf{let } (x = \mathcal{C})\ c & (2)\\
& | & \mathcal{C}\,(\overline{c}) & (3)\\
& | & v\,(\overline{v}, \mathcal{C}, \overline{c}) & (4)\\
& | & \{\ \overline{s : v}, s : \mathcal{C}, \overline{s : c}\ \} & (5)\\
& | & \mathcal{C}\,[c] & (6)\\
& | & v\,[\mathcal{C}] & (7)\\
& | & \mathcal{C}\,[c] = c & (8)\\
& | & v\,[\mathcal{C}] = c & (9)\\
& | & v\,[v] = \mathcal{C} & (10)\\
& | & \textsf{del } \mathcal{C}\,[c] & (11)\\
& | & \textsf{del } v\,[\mathcal{C}] & (12)\\
& | & \textsf{ref } \mathcal{C} & (13)\\
& | & \textsf{deref } \mathcal{C} & (14)\\
& | & \mathcal{C} = c & (15)\\
& | & v = \mathcal{C} & (16)\\
& | & \textsf{if } (\mathcal{C})\ \{\ c\ \}\{\ c\ \} & (17)\\
& | & \mathcal{C};\ c & (18)\\
& | & \textsf{throw } \mathcal{C} & (19)\\
& | & \textsf{break } \ell\ \mathcal{C} & (20)\\
& | & op\,(\overline{v}, \mathcal{C}, \overline{c}) & (21)\\
\mathcal{D} & ::= & [\,] & (1)\\
& | & \textsf{try } \{\ \mathcal{C}[\mathcal{D}]\ \} \textsf{ finally } \{\ c\ \} & (2)\\
& | & \textsf{try } \{\ \mathcal{C}[\mathcal{D}]\ \} \textsf{ catch } (x)\ \{\ c\ \} & (3)\\
& | & \ell : \{\ \mathcal{C}[\mathcal{D}]\ \} & (4)\\
\mathcal{E} & ::= & \mathcal{C}[\mathcal{D}] & (1)
\end{array}$$

Figure 3: Evaluation contexts for $\lambda\rho_{JS}$

Breaks are handled in a similar way, except local contexts are discarded until the matching label are found. In the case of finalizers, the finalization clause is run, followed by reinvoking the break.

The result of a computation is an *answer*, which consists of a store and either a value or error, indicating an uncaught exception:

$$\begin{array}{rcl} A & ::= & \langle \sigma, v \rangle \\ & | & \langle \sigma, \textsf{err } v \rangle \end{array}$$

The *evaluation* of a program is defined by a partial function relating programs to answers:

$$eval(e) = A \text{ if } e \longmapsto\!\!\!\!\!\rightarrow A, \text{ for some } A,$$

where $\longmapsto\!\!\!\!\!\rightarrow$ denotes the reflexive, transitive closure of the standard reduction relation.

2.3 Correspondence with λ_{JS}

We have developed an explicit substitution variant of λ_{JS} in order to derive an environment-based abstract machine,

Environment propagation rules:

$$
\begin{array}{rcl}
(\{\ \overline{s : e}\ \}, \rho) & \rightarrow & \{\ \overline{s : (e, \rho)}\ \} \\
(e(\overline{e}), \rho) & \rightarrow & c(\overline{c}) \\
(\textsf{let}\ (x = e_0)\ e_1, \rho) & \rightarrow & \textsf{let}\ (x = (e_0, \rho))\ (e_1, \rho) \\
(e_0[e_1], \rho) & \rightarrow & (e_0, \rho)[(e_1, \rho)] \\
(e_0[e_1] = e_2, \rho) & \rightarrow & (e_0, \rho)[(e_1, \rho)] = (e_2, \rho) \\
(\textsf{del}\ e_0[e_1], \rho) & \rightarrow & \textsf{del}\ (e_0, \rho)[(e_1, \rho)] \\
(e_0 = e_1, \rho) & \rightarrow & (e_0, \rho) = (e_1, \rho) \\
(\textsf{ref}\ e, \rho) & \rightarrow & \textsf{ref}\ (e, \rho) \\
(\textsf{deref}\ e, \rho) & \rightarrow & \textsf{deref}\ (e, \rho) \\
(\textsf{if}\ (e_0)\ \{\ e_1\ \}\{\ e_2\ \}, \rho) & \rightarrow & \textsf{if}\ ((e_0, \rho))\ \{\ (e_1, \rho)\ \}\{\ (e_2, \rho)\ \} \\
(e_0;\ e_1, \rho) & \rightarrow & (e_0, \rho);\ (e_1, \rho) \\
(\textsf{while}(e_0)\ \{\ e_1\ \}, \rho) & \rightarrow & \textsf{while}((e_0, \rho))\ \{\ (e_1, \rho)\ \} \\
(\ell : \{\ e\ \}, \rho) & \rightarrow & \ell : \{\ (e, \rho)\ \} \\
(\textsf{break}\ \ell\ e, \rho) & \rightarrow & \textsf{break}\ \ell\ (e, \rho) \\
(\textsf{try}\ \{\ e_0\ \}\ \textsf{catch}\ (x)\ \{\ e_1\ \}, \rho) & \rightarrow & \textsf{try}\ \{\ (e_0, \rho)\ \}\ \textsf{catch}\ (x)\ \{\ (e_1, \rho)\ \} \\
(\textsf{try}\ \{\ e_0\ \}\ \textsf{finally}\ \{\ e_1\ \}, \rho) & \rightarrow & \textsf{try}\ \{\ (e_0, \rho)\ \}\ \textsf{finally}\ \{\ (e_1, \rho)\ \} \\
(\textsf{throw}\ e, \rho) & \rightarrow & \textsf{throw}\ (e, \rho) \\
(op(\overline{e}), \rho) & \rightarrow & op(\overline{(e, \rho)})
\end{array}
$$

Context-insensitive, store-insensitive rules:

$$
\begin{array}{rcl}
(x, \rho) & \rightarrow & \rho(x) \\
\textsf{let}\ (x = v)\ (e, \rho) & \rightarrow & (e, \rho[x \mapsto v]) \\
(\textsf{fun}(\overline{x})\ \{\ e\ \}, \rho)(\overline{v}) & \rightarrow & (e, \rho[\overline{x} \mapsto \overline{v}]), \text{ if } |\overline{x}| = |\overline{v}| \\
\{\ \overline{s : v}, s_i : v, \overline{s : v'}\ \}[s_i] & \rightarrow & v \\
\{\ \overline{s : v}\ \}[s_x] & \rightarrow & \textsf{undef}, \text{ if } s_x \notin \overline{s} \\
\{\ \overline{s : v}, s_i : v_i, \overline{s : v'}\ \}[s_i] = v & \rightarrow & \{\ \overline{s : v}, s_i : v, \overline{s : v'}\ \} \\
\{\ \overline{s : v}\ \}[s_x] = v & \rightarrow & \{\ \overline{s : v}\ \}, \text{ if } s_x \notin \overline{s} \\
\textsf{del}\ \{\ \overline{s : v}, s_i : v_i, \overline{s : v'}\ \}[s_i] & \rightarrow & \{\ \overline{s : v}, \overline{s : v'}\ \} \\
\textsf{del}\ \{\ \overline{s : v}\ \}[s_x] & \rightarrow & \{\ \overline{s : v}\ \}, \text{ if } s_x \notin \overline{s} \\
\textsf{if}\ (\textsf{true})\ \{\ c_1\ \}\{\ c_2\ \} & \rightarrow & c_1 \\
\textsf{if}\ (\textsf{false})\ \{\ c_1\ \}\{\ c_2\ \} & \rightarrow & c_2 \\
v;\ c & \rightarrow & c \\
\textsf{while}(c_1)\ \{\ c_2\ \} & \rightarrow & \textsf{if}\ (c_1)\ \{\ c_2;\ \textsf{while}(c_1)\ \{\ c_2\ \}\ \}\{\ \textsf{undef}\ \} \\
\textsf{try}\ \{\ v\ \}\ \textsf{catch}\ (x)\ \{\ c\ \} & \rightarrow & v \\
\textsf{try}\ \{\ v\ \}\ \textsf{finally}\ \{\ c\ \} & \rightarrow & c;\ v \\
\ell : \{\ v\ \} & \rightarrow & v \\
op_n(v_1 \ldots v_n) & \rightarrow & \delta_n(op_n, v_1 \ldots v_n)
\end{array}
$$

Context-sensitive, store-sensitive rules:

$$
\begin{array}{rcl}
\langle \sigma, \mathcal{E}[c] \rangle & \longmapsto & \langle \sigma, \mathcal{E}[c'] \rangle, \text{ if } c \rightarrow c' \\
\langle \sigma, \mathcal{E}[\textsf{ref}\ v] \rangle & \longmapsto & \langle \sigma[a \mapsto v], \mathcal{E}[a] \rangle, \text{ where } a \notin dom(\sigma) \\
\langle \sigma, \mathcal{E}[\textsf{deref}\ a] \rangle & \longmapsto & \langle \sigma, \mathcal{E}[v] \rangle, \text{ if } \sigma(a) = v \\
\langle \sigma, \mathcal{E}[a = v] \rangle & \longmapsto & \langle \sigma[a \mapsto v], \mathcal{E}[v] \rangle \\
\langle \sigma, \mathcal{C}[\textsf{throw}\ v] \rangle & \longmapsto & \langle \sigma, \textsf{err}\ v \rangle \\
\langle \sigma, \mathcal{E}[\textsf{try}\ \{\ \mathcal{C}[\textsf{throw}\ v]\ \}\ \textsf{catch}\ (x)\ \{\ (e, \rho)\ \}] \rangle & \longmapsto & \langle \sigma, \mathcal{E}[(e, \rho[x \mapsto v])] \rangle \\
\langle \sigma, \mathcal{E}[\textsf{try}\ \{\ \mathcal{C}[\textsf{throw}\ v]\ \}\ \textsf{finally}\ \{\ c\ \}] \rangle & \longmapsto & \langle \sigma, \mathcal{E}[c;\ \textsf{throw}\ v] \rangle \\
\langle \sigma, \mathcal{E}[\ell : \{\ \mathcal{C}[\textsf{throw}\ v]\ \}] \rangle & \longmapsto & \langle \sigma, \mathcal{E}[\textsf{throw}\ v] \rangle \\
\langle \sigma, \mathcal{E}[\textsf{try}\ \{\ \mathcal{C}[\textsf{break}\ \ell\ v]\ \}\ \textsf{catch}\ (x)\ \{\ c\ \}] \rangle & \longmapsto & \langle \sigma, \mathcal{E}[\textsf{break}\ \ell\ v] \rangle \\
\langle \sigma, \mathcal{E}[\textsf{try}\ \{\ \mathcal{C}[\textsf{break}\ \ell\ v]\ \}\ \textsf{finally}\ \{\ c\ \}] \rangle & \longmapsto & \langle \sigma, \mathcal{E}[c;\ \textsf{break}\ \ell\ v] \rangle \\
\langle \sigma, \mathcal{E}[\ell : \{\ \mathcal{C}[\textsf{break}\ \ell\ v]\ \}] \rangle & \longmapsto & \langle \sigma, \mathcal{E}[v] \rangle \\
\langle \sigma, \mathcal{E}[\ell' : \{\ \mathcal{C}[\textsf{break}\ \ell\ v]\ \}] \rangle & \longmapsto & \langle \sigma, \mathcal{E}[\textsf{break}\ \ell\ v] \rangle, \text{ if } \ell' \neq \ell
\end{array}
$$

Figure 4: Reduction semantics for $\lambda\rho_{JS}$

which as we will see, is important for the subsequent abstract interpretation. However, let us briefly discuss this new calculus's relation to λ_{JS} and establish their correspondence so that we can rest assured that our analytic framework is really reasoning about λ_{JS} programs.

Our presentation of evaluation contexts for $\lambda\rho_{JS}$ closely follows Guha, *et al.* There are two important differences.

1. The grammar of evaluation contexts for λ_{JS} makes a distinction between local contexts including labels, and local contexts including exception handlers. Let $\mathcal{F}$ and $\mathcal{G}$ denote such contexts, respectively:

$$\mathcal{F} ::= \mathcal{C} \mid \mathcal{C}[\ell : \{\ \mathcal{F}\ \}]$$
$$\mathcal{G} ::= \mathcal{C} \mid \mathcal{C}[\textsf{try}\ \{\ \mathcal{G}\ \}\ \textsf{catch}\ (x)\ \{\ c\ \}].$$

   The distinction allows for exceptions to effectively jump over enclosing labels and for breaks to jump over handlers in one step of reduction:

$$\mathcal{E}[\textsf{try}\ \{\ \mathcal{F}[\textsf{throw}\ v]\ \}\ \textsf{catch}\ (x)\ \{\ (e, \rho)\ \}] \longmapsto \mathcal{E}[(e, \rho[x \mapsto v])],$$

   and

$$\mathcal{E}[\ell' : \{\ \mathcal{G}[\textsf{break}\ \ell\ v]\ \}] \longmapsto \mathcal{E}[v], \text{ if } \ell' = \ell$$
$$\longmapsto \mathcal{E}[\textsf{break}\ \ell\ v], \text{ otherwise.}$$

   It should be clear that our notion of reduction can simulate the above one-step reductions in one or more steps corresponding to the number of labels (exception handlers) in $\mathcal{F}$ (in $\mathcal{G}$). We adopt our single notion of label and handler free local contexts in order to simplify the abstract machine in the subsequent section.

2. The grammar of evaluation of contexts for λ_{JS} mistakenly does not include break contexts in the set of local contexts, causing break expressions within break expression to get stuck, falsifying the soundness theorem. The mistake is minor and easily fixed. When relating $\lambda\rho_{JS}$ to λ_{JS} we assume this correction has been made.

We write λ_{JS} and $\lambda\rho_{JS}$ over a reduction relation to denote the (omitted) one-step reduction relation as given by Guha, *et al.*, corrected as described above, and the one-step reduction as defined in figure 4, respectively.

The results of λ_{JS} and $\lambda\rho_{JS}$ evaluation are related by an unload function $\mathcal{U}$ that recursively applies all of the substitutions represented by an environment, thus mapping a value to a syntactic value. It is the identity function on syntactic values; for answers, functions, and records it is defined as:

$$\mathcal{U}(\langle\sigma, v\rangle) = \langle\sigma, \mathcal{U}(v)\rangle$$
$$\mathcal{U}(\langle\sigma, \textsf{err}\ v\rangle) = \langle\sigma, \textsf{err}\ \mathcal{U}(v)\rangle$$
$$\mathcal{U}(\textsf{fun}(\overline{x})\ \{\ e\ \}, \{(x_0, v_0), \ldots, (x_n, v_n)\}) = \textsf{fun}(x)\ \{\ [\mathcal{U}(v_0)/x_0, \ldots, \mathcal{U}(v_n)/x_n]e\ \}$$
$$\mathcal{U}(\{\ \overline{s : v}\ \}, \{(x_0, v_0), \ldots, (x_n, v_n)\}) = \{\ \overline{s : [\mathcal{U}(v_0)/x_0, \ldots, \mathcal{U}(v_n)/x_n]v}\ \}$$

$$\begin{aligned}
C ::= {} & \mathbf{C}_1 \\
\mid {} & \mathbf{C}_2(C, x, e, \rho) \\
\mid {} & \mathbf{C}_3(C, \overline{c}) \\
\mid {} & \mathbf{C}_4(C, \overline{v}, \overline{c}) \\
\mid {} & \mathbf{C}_5(C, \overline{s : v}, s, \overline{s : c}) \\
\mid {} & \mathbf{C}_6(C, e, \rho) \\
\mid {} & \mathbf{C}_7(C, v) \\
\mid {} & \mathbf{C}_8(C, e, e, \rho) \\
\mid {} & \mathbf{C}_9(C, v, c) \\
\mid {} & \mathbf{C}_{10}(C, v, v) \\
\mid {} & \mathbf{C}_{11}(C, e, \rho) \\
\mid {} & \mathbf{C}_{12}(C, v) \\
\mid {} & \mathbf{C}_{13}(C) \\
\mid {} & \mathbf{C}_{14}(C) \\
\mid {} & \mathbf{C}_{15}(C, e, \rho) \\
\mid {} & \mathbf{C}_{16}(C, v) \\
\mid {} & \mathbf{C}_{17}(C, e, e, \rho) \\
\mid {} & \mathbf{C}_{18}(C, e, \rho) \\
\mid {} & \mathbf{C}_{19}(C) \\
\mid {} & \mathbf{C}_{20}(C, \ell) \\
\mid {} & \mathbf{C}_{21}(C, op, \overline{v}, \overline{c}) \\
D ::= {} & \mathbf{D}_1 \\
\mid {} & \mathbf{D}_2(D, C, e, \rho) \\
\mid {} & \mathbf{D}_3(D, C, x, e, \rho) \\
\mid {} & \mathbf{D}_3(D, C, \ell)
\end{aligned}$$

Figure 5: Continuations

Theorem 1 (Correspondence)**.** *For all programs* e*,*

$$\langle\varnothing, e\rangle \overset{\lambda_{JS}}{\longmapsto\!\!\!\!\!\rightarrow} A \iff \langle\varnothing, (e, \varnothing)\rangle \overset{\lambda\rho_{JS}}{\longmapsto\!\!\!\!\!\rightarrow} A',$$

where $A = \mathcal{U}(A')$*.*

Proof. (Sketch.) The proof follows the structure of Biernacka and Danvy's [4] proof of correspondence for the λ-calculus and Curien's $\lambda\rho$-calculus of explicit substitutions [11], straightforwardly extended to λ_{JS} and $\lambda\rho_{JS}$. □

We have now established our semantic basis: a calculus of explicit substitutions corresponding to λ_{JS}, which is a model adequate for all of JavaScript minus `eval`. In the following section, we apply the syntactic correspondence to derive a correct-by-construction environment-based abstract machine.

3. An abstract machine for JavaScript

In the section, we present the JAM: the JavaScript Abstract Machine.

Evaluation contexts are represented with an algebraic data definition of *continuations*. It suffices to represent only local contexts $\mathcal{C}$ and control contexts $\mathcal{D}$, since a general context $\mathcal{E}$ can be represented by a pair of representations for $\mathcal{C}$ and $\mathcal{D}$ contexts.

$$
\begin{array}{lcl}
\langle \sigma, x, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, D, C, \rho(x)\rangle_{cont} \\
\langle \sigma, k, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, D, C, k\rangle_{cont} \\
\langle \sigma, \mathsf{fun}(\overline{x})\ \{\ e\ \}, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, D, C, (\mathsf{fun}(\overline{x})\ \{\ e\ \}, \rho)\rangle_{cont} \\
\langle \sigma, \{\ \}, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, D, C, \{\ \}\rangle_{cont} \\
\langle \sigma, \{\ s\mathsf{:}e, \overline{s\mathsf{:}e}\ \}, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_5(C, \epsilon, s, \overline{s\mathsf{:}(e,\rho)})\rangle_{eval} \\
\langle \sigma, \mathsf{let}\ (x = e_0)\ e_1, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e_0, \rho, D, \mathbf{C}_2(C, x, e_1, \rho)\rangle_{eval} \\
\langle \sigma, e(\overline{e}), \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_3(C, \overline{(e,\rho)})\rangle_{eval} \\
\langle \sigma, e_0[e_1], \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e_0, \rho, D, \mathbf{C}_6(C, e_1, \rho)\rangle_{eval} \\
\langle \sigma, e_0[e_1] = e_2, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e_0, \rho, D, \mathbf{C}_8(C, e_1, e_2, \rho)\rangle_{eval} \\
\langle \sigma, \mathsf{del}\ e_0[e_1], \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e_0, \rho, D, \mathbf{C}_{11}(C, e_1, \rho)\rangle_{eval} \\
\langle \sigma, e_0 = e_1, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e_0, \rho, D, \mathbf{C}_{15}(C, e_1, \rho)\rangle_{eval} \\
\langle \sigma, \mathsf{ref}\ e, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{13}(C)\rangle_{eval} \\
\langle \sigma, \mathsf{deref}\ e, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{14}(C)\rangle_{eval} \\
\langle \sigma, \mathsf{if}\ (e_0)\ \{\ e_1\ \}\{\ e_2\ \}, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e_0, \rho, D, \mathbf{C}_{17}(C, e_1, e_2, \rho)\rangle_{eval} \\
\langle \sigma, e_0; e_1, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e_0, \rho, D, \mathbf{C}_{18}(C, e_1, \rho)\rangle_{eval} \\
\langle \sigma, \mathsf{while}(e_0)\ \{\ e_1\ \}, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e_0, \rho, D, \mathbf{C}_{17}(C, \mathsf{while}(e_0)\ \{\ e_1\ \}, \mathsf{undef}, \rho)\rangle_{eval} \\
\langle \sigma, \ell\mathsf{:}\{\ e\ \}, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e, \rho, \mathbf{D}_4(D, C, \ell), \mathbf{C}_1\rangle_{eval} \\
\langle \sigma, \mathsf{break}\ \ell\ e, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{20}(C, \ell)\rangle_{eval} \\
\langle \sigma, \mathsf{try}\ \{\ e_0\ \}\ \mathsf{catch}\ (x)\ \{\ e_1\ \}, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e_0, \rho, \mathbf{D}_3(D, C, x, e_1, \rho), \mathbf{C}_1\rangle_{eval} \\
\langle \sigma, \mathsf{try}\ \{\ e_0\ \}\ \mathsf{finally}\ \{\ e_1\ \}, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e_0, \rho, \mathbf{D}_2(D, C, e_1, \rho), \mathbf{C}_1\rangle_{eval} \\
\langle \sigma, \mathsf{throw}\ e, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{19}(C)\rangle_{eval} \\
\langle \sigma, op(), \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, D, C, op()\rangle_{appl} \\
\langle \sigma, op(e, \overline{e}), \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{21}(C, op, \epsilon, \overline{(e,\rho)})\rangle_{eval}
\end{array}
$$

Figure 6: Eval transitions

Continuations represents contexts in an inside-out manner. The grammar of continuations is defined in figure 5. Continuations are constructed to correspond in a one-to-one manner with the productions of evaluation contexts and the constructors have been numbered to correspond with the equation numbers in figure 3.

A local context such as

$$\mathcal{C}[\mathsf{let}\ (x = [\,])\ (e, \rho)],$$

is represented by the continuation $\mathbf{C}_2(C, x, e, \rho)$, where $\mathcal{C}$ is represented by C. Notice that the outermost continuation represents the inner most corresponding context and that closures have been "flattened" within continuations. In some cases, invariants of reduction are reflected in the structure of continuations. For example, when an if-closure is created the branches always have the same environment:

$$\mathsf{if}\ (c_0)\ \{\ (e_1, \rho)\ \}\{\ (e_2, \rho)\ \},$$

which is reflected in the structure of the if-continuation by having only a single environment:

$$\mathcal{C}[\mathsf{if}\ ([\,])\ \{\ (e_1, \rho)\ \}\{\ (e_2, \rho)\ \}]$$

is represented by $\mathbf{C}_{17}(C, e_1, e_2, \rho)$, where $\mathcal{C}$ is represented by C. Control contexts are represented in a similar manner. So for example,

$$\mathcal{D}[\mathcal{C}[\mathsf{try}\ \{\ [\,]\ \}\ \mathsf{catch}\ (x)\ \{\ (e, \rho)\ \}]]$$

is represented by $\mathbf{D}_3(D, C, x, e, \rho)$, where $\mathcal{D}$ and $\mathcal{C}$ are represented by D and C, respectively.

The abstract machine takes the form of a first-order state transition system. There are three classes of transitions for the machine: those that evaluate, those that continue, and those that apply.

evaluate: Evaluation transitions operate over 5-tuples consisting of the store, an expression (the control string), an environment, a control continuation, and a location continuation. The control string is being evaluated within the context represented by the control and local continuation and the eval transitions implement a search for a redex or a value. These transitions dispatch on the expression. If the expression is a syntactic value, then the search for a value has completed, so the machine transitions to a continue state. If the expression is a redex, then the search for a redex has completed, so the machine transitions to an apply state to contract the redex. If the expression is a compound expression, the search for a value or redex continues, so the machine selects the next subexpression to search and pushes a continuation.

Unless the expression is an exception handler, finalizer, or label, a local continuation frame is pushed onto the local continuation. Otherwise, the local continuation is packaged up in a control frame and pushed on the control continuation and a new empty local continuation is installed.

$$
\begin{array}{lcl}
\langle \sigma, \mathbf{D}_1, \mathbf{C}_1, v\rangle_{cont} & \longmapsto & \langle v, \sigma\rangle \\
\langle \sigma, \mathbf{D}_2(D, C, e, \rho), \mathbf{C}_1, v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{22}(C, v)\rangle_{eval} \\
\langle \sigma, \mathbf{D}_3(D, C, x, e, \rho), \mathbf{C}_1, v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, v\rangle_{cont} \\
\langle \sigma, \mathbf{D}_4(D, C, \ell), \mathbf{C}_1, v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, v\rangle_{cont} \\
\langle \sigma, D, \mathbf{C}_2(C, x, e, \rho), v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho[x \mapsto v], D, C\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_3(C, \epsilon), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, v\,\texttt{()}\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_3(C, (e, \rho), \overline{(e, \rho)}), v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_4(C, v, \epsilon, \overline{(e, \rho)})\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_4(C, v_0, \overline{v}, \epsilon), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, v_0\,\texttt{(}\overline{v}, v\texttt{)}\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_4(C, v_0, \overline{v}, (e, \rho), \overline{(e, \rho)}), v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_4(C, v_0, \overline{v}, v, \overline{(e, \rho)})\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_5(C, \overline{s:v}, s, \epsilon), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, \texttt{\{}\ \overline{s:v}, s:v\ \texttt{\}}\rangle_{cont} \\
\langle \sigma, D, \mathbf{C}_5(C, \overline{s:v}, s_0, s:e, \overline{s:e}), v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_5(C, \overline{s:v}, s_0:v, s, \overline{s:e})\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_6(C, e, \rho), v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_7(C, v)\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_7(C, v_0), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, v_0\,\texttt{[}v\texttt{]}\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_8(C, e_0, e_1, \rho), v\rangle_{cont} & \longmapsto & \langle \sigma, e_0, \rho, D, \mathbf{C}_9(C, v, e_1, \rho)\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_9(C, v_0, e, \rho), v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{10}(C, v_0, v)\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_{10}(C, v_0, v_1), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, v_0\,\texttt{[}v_1\texttt{]}\ \texttt{=}\ v\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_{11}(C, e, \rho), v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{12}(C, v)\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_{12}(C, e, v_0), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, \textsf{del}\ v_0\,\texttt{[}v\texttt{]}\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_{13}(C), v\rangle_{cont} & \longmapsto & \langle put(\sigma, a, v), D, C, a\rangle_{cont} \text{ where } a = alloc(\varsigma) \\
\langle \sigma, D, \mathbf{C}_{14}(C), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, \textsf{deref}\ v\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_{15}(C, e, \rho), v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{16}(C, v)\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_{16}(C, v_0), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, v_0\ \texttt{=}\ v\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_{17}(C, e_0, e_1, \rho), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, (\textsf{if}\ \texttt{(}v\texttt{)}\ \texttt{\{}\ e_0\ \texttt{\}\{}\ e_1\ \texttt{\}}, \rho)\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_{18}(C, e, \rho), v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho, D, C\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_{19}(C), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, \textsf{throw}\ v\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_{20}(C, \ell), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, \textsf{break}\ \ell\ v\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_{21}(C, op, \overline{v}, \epsilon), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, op\,\texttt{(}\overline{v}, v\texttt{)}\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_{21}(C, op, \overline{v}, (e, \rho), \overline{(e, \rho)}), v\rangle_{cont} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{21}(C, op, \overline{v}, v, \overline{(e, \rho)})\rangle_{eval} \\
\langle \sigma, D, \mathbf{C}_{22}(C, v_0), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, v_0\rangle_{cont} \\
\langle \sigma, D, \mathbf{C}_{23}(C, v_0), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, \textsf{throw}\ v_0\rangle_{appl} \\
\langle \sigma, D, \mathbf{C}_{24}(C, \ell, v_0), v\rangle_{cont} & \longmapsto & \langle \sigma, D, C, \textsf{break}\ \ell\ v_0\rangle_{appl}
\end{array}
$$

Figure 7: Continue transitions

continue: Continue transitions operate over a 4-tuple consisting of a store, a control and local continuation, and a value. The value is being plugged into the context represented by the continuations, which is what these transitions dispatch upon. If plugging the value into the represented context results in a redex, the machine transitions to an apply state. If plugging the value reveals the next expression that needs to be evaluated, the machine transitions to an evaluation state. If plugging the value in turn results in *another* value, the machine transitions to a continue state to plug that value. Finally, if both the control and local continuation are empty, the answer to the program has been reached and the machine halts.

apply: Apply transitions operate over a 4-tuple consisting of a store, local and control continuation, and a redex. These transitions dispatch on the redex and serve to contract the redex. Since reductions are potentially store- and context-sensitive, the transitions may also dispatch on the control continuation in order to implement the control operators.

The machine relies on three functions for interacting with the store:

$$
\begin{aligned}
alloc &: State \to Address^n \\
put &: Store \times Address \times Value \to Store \\
get &: Store \times Address \to Value
\end{aligned}
$$

The $alloc$ function makes explicit the non-deterministic choice of addresses when allocating space in the store. It returns a vector of addresses, often a singleton, based on the current state of the machine. For the moment, all that we require of $alloc$ is that it return an address that is not in use in the store. The put function updates a store location and is defined as:

$$put(\sigma, a, v) = \sigma[a \mapsto v],$$

and the get function retrieves a value from a store location:

$$get(\sigma, a) = \sigma(a).$$

$$
\begin{array}{lcl}
\langle \sigma, D, C, (\textsf{fun}(\overline{x})\ \{\ e\ \}, \rho)(\overline{v})\rangle_{appl} & \longmapsto & \langle \sigma, e, \rho[\overline{x} \mapsto \overline{v}], D, C\rangle_{eval} \text{ if } |\overline{x}| = |\overline{v}| \\
\langle \sigma, D, C, \{\ \overline{s:v}, s_i:v, \overline{s:v}'\ \}[s_i]\rangle_{appl} & \longmapsto & \langle \sigma, D, C, v\rangle_{cont} \\
\langle \sigma, D, C, \{\ \overline{s:v}\ \}[s_x]\rangle_{appl} & \longmapsto & \langle \sigma, D, C, \textsf{undef}\rangle_{cont} \text{ if } s_x \notin \overline{s} \\
\langle \sigma, D, C, \{\ \overline{s:v}, s_i:v_i, \overline{s:v}'\ \}[s_i] = v\rangle_{appl} & \longmapsto & \langle \sigma, D, C, \{\ \overline{s:v}, s_i:v, \overline{s:v}'\ \}\rangle_{cont} \\
\langle \sigma, D, C, \textsf{del}\ \{\ \overline{s:v}, s_i:v_i, \overline{s:v}'\ \}[s_i]\rangle_{appl} & \longmapsto & \langle \sigma, D, C, \{\ \overline{s:v}, \overline{s:v}'\ \}\rangle_{cont} \\
\langle \sigma, D, C, \textsf{del}\ \{\ \overline{s:v}\ \}[s_x]\rangle_{appl} & \longmapsto & \langle \sigma, D, C, \{\ \overline{s:v}\ \}\rangle_{cont} \text{ if } s_x \notin \overline{s} \\
\langle \sigma, D, C, (\textsf{if}\ (\textsf{true})\ \{\ e_0\ \}\{\ e_1\ \}, \rho)\rangle_{appl} & \longmapsto & \langle \sigma, e_0, \rho, D, C\rangle_{eval} \\
\langle \sigma, D, C, (\textsf{if}\ (\textsf{false})\ \{\ e_0\ \}\{\ e_1\ \}, \rho)\rangle_{appl} & \longmapsto & \langle \sigma, e_1, \rho, D, C\rangle_{eval} \\
\langle \sigma, D, C, op_n(v_1, \ldots, v_n)\rangle_{cont} & \longmapsto & \langle \sigma, D, C, v\rangle_{cont} \text{ if } \delta_n(op_n, v_1, \ldots, v_n) = v \\
\langle \sigma, D, C, \textsf{deref}\ a\rangle_{appl} & \longmapsto & \langle \sigma, D, C, v\rangle_{cont} \text{ if } \sigma(a) = v \\
\langle \sigma, D, C, a = v\rangle_{appl} & \longmapsto & \langle put(\sigma, a, v), D, C, v\rangle_{cont} \\
\langle \sigma, \mathbf{D}_1, C, \textsf{throw}\ v\rangle & \longmapsto & \langle \textsf{err}\ v, \sigma\rangle \\
\langle \sigma, \mathbf{D}_2(D, C, e, \rho), C', \textsf{throw}\ v\rangle_{appl} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{23}(C, v)\rangle_{eval} \\
\langle \sigma, \mathbf{D}_3(D, C, x, e, \rho), C', \textsf{throw}\ v\rangle_{appl} & \longmapsto & \langle \sigma, e, \rho[x \mapsto v], D, C\rangle_{eval} \\
\langle \sigma, \mathbf{D}_4(D, C, \ell), C', \textsf{throw}\ v\rangle_{appl} & \longmapsto & \langle \sigma, D, C, \textsf{throw}\ v\rangle_{appl} \\
\langle \sigma, \mathbf{D}_2(D, C, e, \rho), C', \textsf{break}\ \ell\ v\rangle_{appl} & \longmapsto & \langle \sigma, e, \rho, D, \mathbf{C}_{24}(C, \ell, v)\rangle_{eval} \\
\langle \sigma, \mathbf{D}_3(D, C, x, e, \rho), C', \textsf{break}\ \ell\ v\rangle_{appl} & \longmapsto & \langle \sigma, D, C, \textsf{break}\ \ell\ v\rangle_{appl} \\
\langle \sigma, \mathbf{D}_4(D, C, \ell'), C', \textsf{break}\ \ell\ v\rangle_{appl} & \longmapsto & \langle \sigma, D, C, v\rangle_{cont} \text{ if } \ell' \neq \ell \\
\langle \sigma, \mathbf{D}_4(D, C, \ell'), C', \textsf{break}\ \ell\ v\rangle_{appl} & \longmapsto & \langle \sigma, D, C, \textsf{break}\ \ell\ v\rangle_{appl}
\end{array}
$$

Figure 8: Apply transitions

We make explicit the use of these three functions because they will form the essential mechanism of abstracting the machine in the subsequent section.

The machine also makes use of several "special-purposed" continuations, which allow the machine to optimize some transitions. Those continuations are:

$$
\begin{array}{rcl}
C & ::= & \ldots \\
& | & \mathbf{C}_{22}(C, v) \\
& | & \mathbf{C}_{23}(C, v) \\
& | & \mathbf{C}_{24}(C, \ell, v),
\end{array}
$$

representing $\mathcal{C}[[\ ]\ ;\ v]$, $\mathcal{C}[[\ ]\ ;\ \textsf{throw}\ v]$, and $\mathcal{C}[[\ ]\ ;\ \textsf{break}\ \ell\ v]$.

The initial machine configuration is an evaluation state consisting of the empty store, the program, the empty environment, and the empty control and local continuation:

$$inj(e) = \langle \varnothing, e, \varnothing, \mathbf{D}_1, \mathbf{C}_1\rangle_{eval}.$$

Final configurations are answers just as with standard reduction. The JAM is a correct evaluator for $\lambda\rho_{JS}$.

Theorem 2 (Correctness)**.** *For all programs e,*

$$\langle \varnothing, (e, \varnothing)\rangle \overset{\lambda\rho_{JS}}{\longmapsto\!\!\!\!\!\rightarrow} A \iff inj(e) \overset{JAM}{\longmapsto\!\!\!\!\!\rightarrow} A.$$

Proof. By the correctness of refocusing [13] and the (trivial) meaning preservation of subsequent transformations. □

The detailed step-by-step transformation from the reduction semantics to the abstract machine has been carried out in the meta-language of SML and is included in the supplemental materials.

For the purposes of program analysis, we rely on the following definition of a program's reachable machine states, where ς ranges over states:

$$JAM(e) = \{\varsigma \mid inj(e) \overset{JAM}{\longmapsto\!\!\!\!\!\rightarrow} \varsigma\}.$$

The set $JAM(e)$ is potentially infinite and membership is clearly undecidable. In the next section, we devise a sound approximation to the set $JAM(e)$ by bounding the state-space to a finite set. In the subsequent section, we devise an alternate approximation that approximates $JAM(e)$ by a pushdown automaton.

4. Finite-state abstractions of JavaScript

We make two *concrete* refactorings. We add a level of indirection through the store for both variable bindings and continuations.

4.1 Store-allocated variable bindings

To achieve the first refactoring, we change the environment to no longer map a variable to a value, but instead to map a variable to an *address*, which in turn is used to retrieve a value from the store. Consequently the evaluation transition for variable look-up changes from

$$
\begin{array}{rcl}
\langle \sigma, x, \rho, D, C\rangle_{eval} & \longmapsto & \langle \sigma, D, C, \rho(x)\rangle_{cont} \\
\text{to} & \longmapsto & \langle \sigma, D, C, get(\sigma, \rho(x))\rangle_{cont}
\end{array}
$$

This requires a corresponding change to the rules that create bindings, namely the apply transitions for function application and the continue transition for let-redexes. Function ap-

plication changes from

$$\begin{aligned}\langle \sigma, D, C, (\textsf{fun}\,(\overline{x})\ \{\ e\ \}, \rho)(\overline{v})\rangle_{appl} &= \varsigma \\ &\longmapsto \langle \sigma, e, \rho[\overline{x} \mapsto \overline{v}], D, C\rangle_{eval} \\ \text{to} \ &\longmapsto \langle put(\sigma, \overline{a}, \overline{v}), e, \rho[\overline{x} \mapsto \overline{a}], D, C\rangle_{eval}, \\ &\quad \text{where } \overline{a} = alloc(\varsigma).\end{aligned}$$

Let reduction change from

$$\begin{aligned}\langle \sigma, D, \mathbf{C}_2(C, x, e, \rho), v\rangle_{cont} &= \varsigma \\ &\longmapsto \langle \sigma, e, \rho[x \mapsto v], D, C\rangle_{eval} \\ \text{to} \ &\longmapsto \langle put(\sigma, a, v), e, \rho[x \mapsto a], D, C\rangle_{eval}, \\ &\quad \text{where } a = alloc(\varsigma).\end{aligned}$$

4.2 Store-allocated continuations

To achieve the second refactoring, we change the constructors for continuations. In each $\mathbf{C}_i$ and $\mathbf{D}_j$ constructor, we replace each continuation with a pointer. So for example, $\mathbf{C}_2(C, x, e, \rho)$ is replaced with the following constructor, $\mathbf{C}_2(a, x, e, \rho)$, and $\mathbf{D}_3(D, C, \ell)$ is replaced by $\mathbf{D}_3(a, a, \ell)$, etc. Machine transitions are altered to so that those that push continuations must first allocate space in the store. Those that pop continuations must dereference space in the store. As an example of a transition that pushes a continuation, consider the evaluation transition for ref expressions

$$\langle \sigma, \textsf{ref}\ e, \rho, D, C\rangle_{eval} \longmapsto \langle \sigma, e, \rho, D, \mathbf{C}_{13}(C)\rangle_{eval}.$$

Since the $\mathbf{C}_{13}(C)$ has been changed to $\mathbf{C}_{13}(a)$ to take an address rather than a continuation, this transition is changed to allocate space pointing to C and the allocated address is used in the continuation, *i.e.*, the transition becomes:

$$\begin{aligned}\langle \sigma, \textsf{ref}\ e, \rho, D, C\rangle_{eval} &= \varsigma \\ &\longmapsto \langle put(\sigma, a, C), e, \rho, D, \mathbf{C}_{13}(a)\rangle_{eval}, \\ &\quad \text{where } a = alloc(\varsigma).\end{aligned}$$

As an example of a continuation that pops a continuation, consider:

$$\langle \sigma, D, \mathbf{C}_{18}(C, e, \rho), v\rangle_{cont} \longmapsto \langle \sigma, e, \rho, D, C\rangle_{eval}.$$

Since $\mathbf{C}_{18}(C, e, \rho)$ has been changed to $\mathbf{C}_{18}(a, e, \rho)$, the revised transition does not immediately have C at hand, however, since pushed continuations have been allocated in the store, we simply dereference a to obtain C. The resulting transitions is:

$$\langle \sigma, D, \mathbf{C}_{18}(a, e, \rho), v\rangle_{cont} \longmapsto \langle \sigma, e, \rho, D, get(\sigma, a)\rangle_{eval},$$

Transitions that pop control continuations are handled similarly; the only difference is that two dereferences are needed. So for example,

$$\begin{aligned}&\langle \sigma, \mathbf{D}_4(D, C, \ell'), C', \textsf{break}\ \ell\ v\rangle_{appl} \\ &\quad \longmapsto \langle \sigma, D, C, \textsf{break}\ \ell\ v\rangle_{appl}\end{aligned}$$

becomes

$$\begin{aligned}&\langle \sigma, \mathbf{D}_4(a_0, a_1, \ell'), C', \textsf{break}\ \ell\ v\rangle_{appl} \\ &\quad \longmapsto \langle \sigma, get(\sigma, a_0), get(\sigma, a_1), \textsf{break}\ \ell\ v\rangle_{appl},\end{aligned}$$

Transitions that both pop and push continuations need to both allocate and dereference continuations in the store.

We call this pointer-refined machine JAM^* and define the reachable states of this machine:

$$JAM^*(e) = \{\varsigma \mid inj(e) \xmapsto{JAM^*}\!\!\!\!\!\rightarrow\!\!\!\!\!\rightarrow \varsigma\}.$$

It should be clear that these two machine transformations are meaning preserving and that resulting machine operates in lock step with the original abstract machine.

Lemma 1. $JAM(e) \simeq JAM^*(e)$

Proof. By induction on the length of the machine trace up to a reachable state. Both machines make matching transitions and the states are easily related through the pointer refinement. □

4.3 Bounding the state-space

Although the JAM and JAM* are isomorphic, the JAM* is in a position to be approximated in a straightforward way that is transparently sound.

The machine's state-space is bounded simply by restricting the set of allocatable addresses to a fixed set of finite size, $\widehat{Address}$. This necessitates a change in the machine transition system and the representation of states. The machine can no longer restrict allocated addresses to be fresh with respect to the domain of the store as is the case when bindings are allocated, **ref** -expression are evaluated, and continuations are pushed. Instead, the machine calls an allocation function that returns a member of the finite set of addresses. Since the allocation function may return an address already in use, the behavior of the store must change to accommodate multiple values residing in a given location. We let $\hat{\sigma}$ range over such stores:

$$\hat{\sigma} \in \widehat{Store} = \widehat{Address} \to_{\text{fin}} \mathcal{P}(Value) + \mathcal{P}(Continuation).$$

We let $\widehat{JAM^*}$ denote the abstract machine that results from replacing all occurrences of the functions $alloc$, put, and get, with the following counterparts:

$$\begin{aligned}&\widehat{alloc} : State \to \widehat{Address}^n \\ &\widehat{put} : \widehat{Store} \times \widehat{Address} \times Value \to \widehat{Store} \\ &\widehat{get} : \widehat{Store} \times \widehat{Address} \times (Value + Continuation)\end{aligned}$$

The $\widehat{alloc}$ function works like $alloc$, but produces addresses from the finite set $\widehat{Address}$. The $\widehat{put}$ function updates a store location by *joining* the given value to any existing values that reside at that address:

$$\widehat{put}(\hat{\sigma}, a, v) = \sigma[a \mapsto \{v\} \cup \hat{\sigma}(a)].$$

The $\widehat{get}$ relation is interpreted as a non-deterministic choice of an element at the given location; $\widehat{get}(\hat{\sigma}, a)$ denotes a value or continuations x such that $x \in \hat{\sigma}(a)$.

We can formally relate the JAM* to its abstracted counterpart through the natural structural abstraction map α on their state-spaces. This map recurs over the state-space of the JAM* to inflict a finitizing abstraction only at the leaves its state-space—addresses and primitive values—and structures that cannot soundly absorb that finitization, which in this case, is only the store. The range of the store expands into a power set, so that when an abstract address is *re*-allocated, it can hold both the existing values and the newly added value; formally:

$$\alpha(\sigma) = \lambda\hat{a}. \bigsqcup_{\alpha(a)=\hat{a}} \alpha(\sigma(a)).$$

Guha *et al.* handle primitive operations in λ_{JS} in standard fashion by delegating to a δ-function. For the sake of analysis, we can delegate to any sound, finite abstraction of the δ-function. The simplest such abstraction maps values to their types, which makes the abstract δ function isomorphic to its intensional signature. For in-depth discussion of richer abstract domains over basic values for use in JavaScript, we refer the reader to Jensen *et al.* [20]; they provide abstract domains for JavaScript which could be plugged directly into the $\widehat{JAM^*}$.

The abstracted JAM* provides a sound simulation of the JAM* and, by lemma 1 and theorems 1 and 2, a sound simulation of the JAM, $\lambda\rho_{JS}$, and λ_{JS}, as well.

Theorem 3 (Soundness)**.** *If* $\varsigma \overset{JAM^*}{\longmapsto} \varsigma'$ *and* $\alpha(\varsigma) \sqsubseteq \hat{\varsigma}$*, then there exists an abstract state* $\hat{\varsigma}'$*, such that* $\hat{\varsigma} \overset{\widehat{JAM^*}}{\longmapsto} \hat{\varsigma}'$ *and* $\alpha(\varsigma') \sqsubseteq \hat{\varsigma}'$*.*

Proof. We reason by case analysis on the transition. In the cases where the transition is deterministic, the result follows by calculation. For the the remaining non-deterministic cases, we must show an abstract state exists such that the simulation is preserved. By examining the rules for these cases, we see that all hinge on the abstract store in $\hat{\varsigma}$ soundly approximating the concrete store in ς, which follows from the assumption that $\alpha(\varsigma) \sqsubseteq \hat{\varsigma}$. □

We have now established the abstracted JAM* is sound, but it is also straightforward to observe its state-space is finite:

$$\widehat{JAM^*}(e) = \{\hat{\varsigma} \mid inj(e) \overset{\widehat{JAM^*}}{\twoheadrightarrow} \hat{\varsigma}\}.$$

The finiteness of the state-space leads to the straightforward observation of decidability for reachability-based program analysis questions.

Theorem 4 (Decidability)**.** $\hat{\varsigma} \in \widehat{JAM^*}(e)$ *is decidable.*

Proof. The state-space of the machine is non-recursive with finite sets at the leaves on the assumption that addresses are finite. Hence reachability is decidable since the abstract state-space is finite. □

5. Pushdown abstractions of JavaScript

Pushdown analysis is an alternative paradigm for the analysis of higher-order programs in which the run-time program stack is precisely modeled with the stack of a pushdown system. Consequently, a pushdown analysis can exactly match control flow transfers from calls to returns, from throws to handlers, and from breaks to labels. This in contrast with the traditional approaches of finite-state abstractions which necessarily model the control stack with finite bounds.

As an example demonstrating the basic difference between traditional approaches, such as 0CFA, and our pushdown approach, consider the following JavaScript program:

```
// (ℝ → ℝ) → (ℝ → ℝ)
// Compute an approximate derivative of f.
function deriv(f) {
  var ϵ = 0.0001;
  return function (x) {
    return (f(x+ϵ) - f(x-ϵ)) / (2*ϵ);
  };
};
deriv(function (y) { return y*y; });
```

The `deriv` program computes an approximation to the derivative of its argument. In this example, it is being applied the square function, so it returns an approximation to the double function.

It is important to take note of the two distinct calls to `f`. Basic program analyses, such as 0CFA, will determine that the square function is the target of the call at `f(x+ϵ)`. *However, they cannot determine whether the call to* `f(x+ϵ)` *should return to* `f(x+ϵ)` *or to* `f(x-ϵ)`. Context-sensitive analysis, such as 1CFA, can reason more precisely by distinguishing the analysis of each call to `f`, however such methods come with a prohibitive computational cost [38] and, more fundamentally, k-CFA will only suffice for precisely reasoning about the control stack up to a fixed calling context depth.

This is the fundamental shortcoming of traditional approaches to higher-order program analysis, both in functional and object-oriented languages. This is an unfortunate situation, since the dominate control mechanism is calls and returns. To make matters worse, in addition to higher-order functions, JavaScript includes sophisticated control mechanisms further complicating and confounding analytic approaches.

To overcome this shortcoming we propose a *pushdown* approach to abstraction that exactly captures the behavior of the control stack. As in section 4, we derive the pushdown analysis as an abstract interpretation of the JAM machine. The main difference between the approach of this section and previous is that we will leave the stack component

untouched and unbounded. As this abstract interpretation ranges over an infinite state-space, we the main technical difficult will be recovering decidability of reachable states.

The basic idea is to repeat the refactoring of section 4.1, moving variable bindings into the store, but to avoid the refactoring of section 4.2 that threads continuations through the store. Instead we will leave continuations on the stack, then bound the store as in section 4.3.

The idea is that by bounding the store, the control stack, while unbounded, will consist of a finite stack alphabet. Since the remaining components of the machine are finite, the abstract machine is equivalent in computational power to a pushdown automaton, and thus reachability questions are casts naturally in terms of decidable PDA reachability properties.

Unfortunately, there is an immediate problem with the described approach when applied to the JavaScript abstract machine. The problem stems from the JAM having two control stacks. Consequently, when abstracting we arrive at a two-stack pushdown machine, which in general has the power to simulate a Turing-machine. However this problem can be overcome: the JAM can be reformulated into a single stack machine in such a way that preserves correctness and enables a pushdown abstraction that is decidable.

5.1 Reformulation of reduction semantics

One of the lessons of our abstract machine-based approach to analysis is that many problems in program analysis can be solved at the semantic level and then imported systematically to the analytic side. So for example, abstract garbage collection [28] can be expressed as concrete garbage collection with the pointer refinement and store abstraction of section 4 applied [39]. Similarly, the exponential complexity of k-CFA can be avoided by concretely changing the representation of closures and then abstracting in an unremarkable way [29].

We likewise solve our two-stack problem by a reformulation at the level of the reduction semantics for $\lambda\rho_{JS}$ and then repeat the refocusing construction to derive a one-stack variant of the JAM.

The basic reason for maintaining the control and local stack is to allow jumps over the local context whenever a control operator is invoked. This is seen in the reduction semantics with reductions such as this:

$$\begin{aligned}\mathcal{E}[\ell' : \{\ \mathcal{C}[\textsf{break}\ \ell\ v]\ \}] &\longmapsto \mathcal{E}[v] \text{ if } \ell' = \ell \\ &\longmapsto \mathcal{E}[\textsf{break}\ \ell\ v] \text{ if } \ell' \neq \ell\end{aligned}$$

To enable a single stack, we simulate this jump over the local context by "bubbling" over it in a piecemeal fashion. This is accomplished by defining a notion of single, non-empty local context frames $\mathcal{S}$, *i.e.*,

$$\begin{aligned}\mathcal{S} ::= &\ \textsf{let}\ (x = [\,])\ c \\ \mid &\ [\,](\overline{c}) \\ \mid &\ c(\overline{v}, [\,], \overline{c}) \\ &\ \vdots \\ \mid &\ \textsf{break}\ \ell\ [\,] \\ \mid &\ op(\overline{v}, [\,], \overline{c})\end{aligned}$$

The reduction relation for control operators remains context sensitive, but does not operate over whole contexts, but just the enclosing frame, which can then be implemented with a stack. The rules for simulating the above reduction are then:

$$\begin{aligned}\mathcal{E}[\mathcal{S}[\textsf{break}\ \ell\ v]] &\longmapsto \mathcal{E}[\textsf{break}\ \ell\ v] \\ \mathcal{E}[\ell' : \{\ \textsf{break}\ \ell\ v\ \}] &\longmapsto \mathcal{E}[v] \text{ if } \ell' = \ell \\ &\longmapsto \mathcal{E}[\textsf{break}\ \ell\ v] \text{ if } \ell' \neq \ell\end{aligned}$$

Clearly, these reductions simulate the original single reduction by a number of steps that corresponds to the number of local frames between the label and the break.

The complete replacement of the context-sensitive reductions is given in figure 9. We refer to this alternative reduction semantics as $\lambda\rho'_{JS}$.

Lemma 2. *For all programs e,*

$$\langle \varnothing, (e, \varnothing)\rangle \xmapsto{\lambda\rho_{JS}}\!\!\!\!\!\rightarrow A \iff \langle \varnothing, (e, \varnothing)\rangle \xmapsto{\lambda\rho'_{JS}}\!\!\!\!\!\rightarrow A.$$

5.2 A single stack machine for JavaScript

For our single stack variant of the JAM, which we refer to as the JAM$'$, the representation of continuations is revised to unify C and D continuations as follows:

$$\begin{aligned}E ::= &\ \mathbf{E}_1 \\ \mid &\ \mathbf{C}'_2(E, x, e, \rho) \\ \mid &\ \mathbf{C}'_3(E, \overline{c}) \\ &\ \vdots \\ \mid &\ \mathbf{C}'_{21}(E, op, \overline{v}, \overline{c}) \\ \mid &\ \mathbf{D}'_2(E, e, \rho) \\ \mid &\ \mathbf{D}'_3(E, x, e, \rho) \\ \mid &\ \mathbf{D}'_3(E, \ell)\end{aligned}$$

Notice that E includes all of the C and D constructors, re-interpreted to contain E subcontinuations. We use E to denote the single empty continuation.

The injection function for the initial machine state is defined as:

$$inj'(e) = \langle \varnothing, e, \varnothing, \mathbf{E}_1\rangle_{eval}$$

The pair of D and E continuations of each machine state is collapsed into a single E component and the machine transitions are revised as follows:

evaluate: Whenever an evaluation transition of the JAM pushed a continuation either on to the control or local continuation, the JAM$'$ now pushes the corresponding continuation on to E.

$$
\begin{array}{rcl}
\langle \sigma, \textsf{throw } v\rangle & \to & \langle \sigma, \textsf{err } v\rangle \\
\langle \sigma, \mathcal{E}[\mathcal{S}[\textsf{throw } v]]\rangle & \to & \langle \sigma, \mathcal{E}[\textsf{throw } v]\rangle \\
\langle \sigma, \mathcal{E}[\textsf{try \{ throw } v \textsf{ \} catch (}x\textsf{) \{ }(e,\rho)\textsf{ \}}]\rangle & \to & \langle \sigma, \mathcal{E}[(e, \rho[x \mapsto v])]\rangle \\
\langle \sigma, \mathcal{E}[\textsf{try \{ throw } v \textsf{ \} finally \{ }c\textsf{ \}}]\rangle & \to & \langle \sigma, \mathcal{E}[c\textsf{; throw } v]\rangle \\
\langle \sigma, \mathcal{E}[\ell\textsf{:\{ throw } v \textsf{ \}}]\rangle & \to & \langle \sigma, \mathcal{E}[\textsf{throw } v]\rangle \\
\langle \sigma, \mathcal{E}[\mathcal{S}[\textsf{break } \ell\ v]]\rangle & \to & \langle \sigma, \mathcal{E}[\textsf{break } \ell\ v]\rangle \\
\langle \sigma, \mathcal{E}[\textsf{try \{ break } \ell\ v \textsf{ \} catch (}x\textsf{) \{ }c\textsf{ \}}]\rangle & \to & \langle \sigma, \mathcal{E}[\textsf{break } \ell\ v]\rangle \\
\langle \sigma, \mathcal{E}[\textsf{try \{ break } \ell\ v \textsf{ \} finally \{ }c\textsf{ \}}]\rangle & \to & \langle \sigma, \mathcal{E}[c\textsf{; break } \ell\ v]\rangle \\
\langle \sigma, \mathcal{E}[\ell\textsf{:\{ break } \ell\ v \textsf{ \}}]\rangle & \to & \langle \sigma, \mathcal{E}[v]\rangle \\
\langle \sigma, \mathcal{E}[\ell'\textsf{:\{ break } \ell\ v \textsf{ \}}]\rangle & \to & \langle \sigma, \mathcal{E}[\textsf{break } \ell\ v]\rangle, \text{ if } \ell' \neq \ell
\end{array}
$$

Figure 9: Reformulated reduction semantics

continue: The continue transitions of the JAM either dispatched on the local continuation or the control continuation and the empty local continuation. In the case of the local continuation, the transitions are rewritten to correspondingly dispatch on E.

In the case of the control context, whenever the E is a control frame, there is implicitly an empty local continuation between the value and that frame, so the JAM$'$ makes the same transitions. The only transition that needs special attention is:

$$\langle e, \mathbf{D}_1, \mathbf{C}_1, v\rangle_{cont} \longmapsto \langle v, \sigma\rangle, \text{ which becomes}$$
$$\langle e, \mathbf{E}_1, v\rangle_{cont} \longmapsto \langle v, \sigma\rangle.$$

apply: The application transitions are rewritten in a straightforward way to implement the reformulated reduction relation as given in figure 9.

Theorem 5 (Correctness)**.** *For all programs e,*

$$\langle \varnothing, (e, \varnothing)\rangle \overset{\lambda\rho_{JS}}{\longmapsto\!\!\!\!\!\rightarrow} A \iff inj'(e) \overset{JAM'}{\longmapsto\!\!\!\!\!\rightarrow} A.$$

5.3 Bounding the store, not the stack

$$\widehat{JAM}'(e) = \{\hat{\varsigma} \mid inj'(e) \overset{\widehat{JAM}'}{\longmapsto\!\!\!\!\!\rightarrow} \hat{\varsigma}\}.$$

Theorem 6 (Soundness)**.** *If* $\varsigma \overset{JAM'}{\longmapsto} \varsigma'$ *and* $\alpha(\varsigma) \sqsubseteq \hat{\varsigma}$*, then there exists an abstract state* $\hat{\varsigma}'$*, such that* $\hat{\varsigma} \overset{\widehat{JAM}'}{\longmapsto} \hat{\varsigma}'$ *and* $\alpha(\varsigma') \sqsubseteq \hat{\varsigma}'$*.*

The proof follows the same structure as that of theorem 3, and is in fact *simplified* since the continuations frames of the machines exactly correspond, eliminating the need to consider the non-deterministic choice of a continuation residing at a store location.

The more interesting aspect of the pushdown abstraction is decidability. Notice that since the stack has a recursive, unbounded structure, the state-space of the machine is potentially infinite so deciding reachability by enumerating the reachable states will no longer suffice.

Theorem 7 (Decidability)**.** $\hat{\varsigma} \in \widehat{JAM}'(e)$ *is decidable.*

Proof. States of the abstracted single-stack JavaScript abstract machine may consist of a store, an expression, a value, and a continuation. Observe that with the exception of continuations, each of these sets is finite: for a given program, there are a fixed set of expressions; environments are finite since they map variables to addresses and the address space is bounded; since expressions and environments are finite, so too are the set of values; stores are finite since addresses and values are finite.

Continuations implement a stack structure and machine transitions dispatch only on the top element and either push or pop a continuation frame at a time, *i.e.*, the machine's use of continuations obeys a stack discipline. The alphabet of the stack consists of continuation tags, $\mathbf{C}'_i$ or $\mathbf{D}'_j$, and a number of expressions, a value, or an environment, all of which are finite sets. Thus the stack alphabet is finite and consequently the machine is a pushdown automaton and decidability follows from known results on pushdown automata. □

6. Validation

In addition to proving our formal claims, we have developed executable models and test beds to validate our results. We have developed the reduction semantics of $\lambda\rho_{JS}$ and $\lambda\rho'_{JS}$ in Standard ML (SML) and carried out the refocusing construction and subsequent program transformations in a step-by-step manner closely following the lecture notes of Danvy [12]. We found using SML as a metalanguage helpful due to its type system and non-exhaustive and redundant pattern matching warnings. For example, we were able to encode Guha *et al.*'s soundness theorem, which is false without the modification to the semantics as described in section 2.3, in SML in such a way that the type of the one-step reduction relation, coupled with exhaustive pattern matching, implies a program is either a value or can make progress.

We ported our semantics and concrete machines to PLT Redex [15] and then built their abstractions. This was done because PLT Redex supports programming with relations and includes a property-based random testing mechanism. The support for programming with relations is an important

aspect for building the non-deterministic transition systems of the abstracted JAM machines since, unlike their concrete counterparts, the transition system cannot be encoded as a function in a straightforward way. Using the random testing framework [23], we tested the correspondence, correctness, and soundness theorems. As an added benefit, we were able to visualize our test programs' state-spaces using the included graphical tools and we leveraged PLT Redex as a fixed-point calculator for the finite-state abstractions since it includes library functions for computing the transitive closure of arbitrary reduction relations.

Finally, we used Guha *et al.*'s code for desugaring in order to test our framework on real JavaScript code. We tested against the same test bed as Guha *et al.*: a significant portion of the Mozilla JavaScript test suite; about 5,000 lines of unmodified code. We tested the closure-based semantics of $\lambda\rho_{JS}$ for correspondence against the substitution-based semantics of λ_{JS} and tested the machines for correctness with respect to the $\lambda\rho_{JS}$ semantics. Finally, we tested the instantiations of our analytic framework for soundness with respect to the machines. Since the semantics of λ_{JS} have been validated against the output of Rhino, V8, and SpiderMonkey, and all of semantic artifacts simulate or approximate λ_{JS}, these tests substantiate our frameworks correctness.

7. Related work

Our approach fits cleanly within the progression of work in abstract interpretation [9, 10] and is inspired by the pioneering work on higher-order program analysis by Jones [22]. Like Jones, our work centers around machine-based abstractions of higher-order languages; and like Jones [35], we have obtained our machines by program transformations of high-level semantic descriptions in the tradition of Reynolds [33]. We have been able to leverage the refocusing approach of Danvy, *et al.*, to systematically derive such machines [13, 4, 12], and our main technical insight has been that threading bindings and continuations through the store results in straightforward and transparently sound framework for finite-state abstractions, while leaving continuations on the stack results in pushdown abstractions that precisely reason about control flow in face of higher-order functions and sophisticated control operators.

7.1 Pushdown analyses

The closest related work for this is Vardoulakis and Shivers recent work on CFA2 [40]. CFA2 is a table-driven summarization algorithm that exploits the balanced nature of calls and returns to improve return-flow precision in a control-flow analysis. Though CFA2 alludes to exploiting context-free languages, context-free languages are not explicit in its formulation in the same way that pushdown systems are in pushdown control-flow analysis [14]. With respect to CFA2, pushdown control-flow analysis is polyvariant, covers direct-style, and the monovariant instantiation is lower in complexity (CFA2 is exponential-time).

On the other hand, CFA2 distinguishes stack-allocated and store-allocated variable bindings, whereas our formulation of pushdown control-flow analysis does not and allocates all bindings in the store. If CFA2 determines a binding can be allocated on the stack, that binding will enjoy added precision during the analysis and is not subject to merging like store-allocated bindings.

This work also draws on CFL- and pushdown-reachability analysis [5, 24, 31, 32]. CFL-reachability techniques have also been used to compute classical finite-state abstraction CFAs [26] and type-based polymorphic control-flow analysis [30]. *These analyses should not be confused with pushdown control-flow analysis*, which is computing a fundamentally more precise kind of CFA. Moreover, Rehof and Fahndrich's method is cubic in the size of the *typed* program, but the types may be exponential in the size of the program. In addition, our technique is not restricted to typed programs.

7.2 JavaScript analyses

Thiemann [37] develops a type system for Core JavaScript, a restricted subset of JavaScript. The type system rules out the application of non-functions, applying primitive operations to values of incorrect base type, dereferencing fields of the `undefined` or `null` value, and referring to unbound variables. The most closely related work to that of section 4 is that of Jensen, Møller, and Thiemann, [20], which takes the form of an abstract interpretation computing type inference. It builds on the type system of Thiemann [37], using it as inspiration for their abstract domains. To improve precision, they incorporate abstract garbage collection. Richards *et al.*'s landmark empirical survey of JavaScript code [34] made it clear that for JavaScript analyses to work in the wild, it is not sufficient to handle only a well-behaved core of the language. Capturing ill-behaved parts of JavaScript soundly and precisely was a major motivation for our research.

Subsequently, Heidegger and Thiemann have extended the type system with a notion of *recency* to improve precision [18] and Jensen *et al.*, have developed a technique of *lazy propagation* to increase the feasibility of the analysis [21]. Balakrishnan and Reps pioneered the idea of recency [3]: objects are considered recent when created and given a singleton type and treated flow-sensitively until "demoted" to a summary type that is treated flow-insensitively. Recency enables strong update in analyses [19], which is important for reasoning precisely about initialization patterns in JavaScript programs. Recency and lazy propagation are orthogonal to our analytic framework: in our recent work, we show how to incorporate a generalization of recency into a machine-based static analysis [27] through the concept of anodization.

Guha, Krishnamurthi and Jim [16] developed an analysis for intrusion-detection of JavaScript, driven in part by

an adaptation of k-CFA to a large subset of JavaScript. Our work differs their work in that we are formally guaranteeing soundness with respect to a concrete semantics, we provide fine-grained control over precision for our finite-state analysis and we also provide a pushdown analysis for handling the complex non-local control features which pervade JavaScript code. (Guha *et al.* make a best-effort attempt at soundness and dynamically detect violations of soundness in empirical trials, violations which they use to refine their analysis.)

Chugh *et al.* [6] present a staged information-flow analysis of JavaScript. In effect, their algorithm partially evaluates the analysis with respect to the available JavaScript to produce a residual analysis. When more code becomes available, the residual analysis resumes. Our own framework is directly amenable to such partial evaluation for handling constructs like `eval`: explore the state-space aggressively, but do not explore past `eval` states. The resulting partial abstract transition graph is sound until the program encounters `eval`. At this point, the analysis may be resumed with the code supplied to `eval`.

8. Conclusions and perspective

We present a principled systematic derivation of machine-based analysis for JavaScript. By starting with an established formal semantics and transforming it into an abstract machine, we soundly capture JavaScript in full, quirks and all. The abstraction of this machine yields a robust finite-state framework for the static analysis of JavaScript, capable of instantiating the equivalent of traditional techniques such as k-CFA and CPA. Finding the traditional finite-state approach wanting in precision for JavaScript's extensive use of non-local control, we extend the theory of systematic abstraction of abstract machines from finite-state to pushdown. These decidable pushdown machines precisely model the structure of the program stack, and so do not lose precision in the presence of control constructs that depend on it, such as recursion or complex exceptional control-flow.

`http://lambda-calcul.us/jam/`

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